

Two stage design for estimating the product of means with cost in the case of the exponential family

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Abstract

We investigate the problem of estimating the product of means of independent populations from the one parameter exponential family in a Bayesian framework. We give a random design which allocates m_i the number of observations from population P_i such that the Bayes risk associated with squared error loss and cost per unit observation is as small as possible. The design is shown to be asymptotically optimal.

Keywords: Two stage design, Exponential family, Bayes risk, Cost, Asymptotic optimality.

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1. Introduction

Assume that for $i = 1, \dots, n$; a random variable X_i whose distribution belongs to the one parameter exponential family is observable from population P_i with cost c_i per unit observation. The problem of interest is to estimate the product of means using a Bayesian approach associated with squared error loss and cost. Since a Bayesian framework is considered, cf. [1, 5], then typically optimal estimators are Bayesian estimators and the problem turns to design a sequential allocation scheme, cf. [10], to select m_i the number of observations from population P_i such that the Bayes risk plus the corresponding budget $B = \sum_{i=1}^n c_i m_i$ is as small as possible. In [6],

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a sequential design was defined to estimate the difference between means of two populations from the exponential family with associated cost. The random allocation was shown to be the best from numerical considerations, cf. [7, 8, 9]. Similarly, the problem of estimating the product of several means of independent populations, subject to the constraint of a total number of observations M fixed, was addressed in [2] using a two stage approach. The allocation of m_i was nonrandom and the first order optimality was shown for large M , cf. [3, 4].

Suppose that X_i has the distribution of the form

$$f_{\theta_i}(x_i) = e^{\theta_i x_i - \psi(\theta_i)}, \quad x_i \in \mathbb{R}, \quad \theta_i \in \Omega$$

where Ω is a bounded open interval in \mathbb{R} . It follows that $E_{\theta_i}[x_i] = \psi'(\theta_i)$ and $Var_{\theta_i}[x_i] = \psi''(\theta_i)$. Our aim is to estimate the product

$$\theta = \prod_{i=1}^n \psi'(\theta_i)$$

subject to squared error loss and cost. One assumes that prior distribution for each θ_i is given by

$$\pi_i(\theta_i) = \frac{e^{r_i \mu_i \theta_i - r_i \psi(\theta_i)}}{\int_{\Omega} e^{r_i \mu_i \theta_i - r_i \psi(\theta_i)} d\theta_i}$$

where r_i and μ_i are reals and $r_i > 0$, $i = 1, \dots, n$. Here we treat θ_i as a realization of a random variable and assume that for each population, x_{i1}, \dots, x_{im_i} are conditionally independent and that $\theta_1, \dots, \theta_n$ are a priori independent.

2. The Bayes risk

Posterior distributions of θ_i are given by

$$\pi_i(\theta_i/X_i) = \pi_i(\theta_i/x_{i1}, \dots, x_{im_i}) = \frac{e^{r_{m_i} \mu_{m_i} \theta_i - r_{m_i} \psi(\theta_i)}}{\int_{\Omega} e^{r_{m_i} \mu_{m_i} \theta_i - r_{m_i} \psi(\theta_i)} d\theta_i}$$

where $r_{m_i} = r_i + m_i$ and

$$\mu_{m_i} = \frac{r_i}{m_i + r_i} \mu_i + \frac{\sum_{j=1}^{m_i} x_{ij}}{m_i + r_i}$$

Let us denote by $\mathcal{F}_{m_1, \dots, m_n}$ the σ -Field generated by (X_1, \dots, X_n) where $X_i = (x_{i1}, \dots, x_{im_i})$ and let $\mathcal{F}_{m_i} = \sigma(X_i) = \sigma(x_{i1}, \dots, x_{im_i})$. It was shown in [8] that

$$E[\psi'(\theta_i) / \mathcal{F}_{m_i}] = \mu_{m_i} \quad (1)$$

$$Var[\psi'(\theta_i) / \mathcal{F}_{m_i}] = E\left[\frac{\psi''(\theta_i)}{m_i + r_i} / \mathcal{F}_{m_i}\right] \quad (2)$$

Using independence across populations, the Bayes estimator of θ is

$$\hat{\theta} = E[\theta / \mathcal{F}_{m_1, \dots, m_n}] = \prod_{i=1}^n E[\psi'(\theta_i) / \mathcal{F}_{m_i}]$$

Hence, the posterior expected loss is

$$l(m_1, \dots, m_n) = E\left[\prod_{i=1}^n \psi'^2(\theta_i) / \mathcal{F}_{m_1, \dots, m_n}\right] - E^2\left[\prod_{i=1}^n \psi'(\theta_i) / \mathcal{F}_{m_1, \dots, m_n}\right]$$

and with the help of independence, equations (1), (2) and the fact that

$$E^2[\psi'(\theta_i) / \mathcal{F}_{m_i}] = Var[\psi'(\theta_i) / \mathcal{F}_{m_i}] - E[\psi'^2(\theta_i) / \mathcal{F}_{m_i}]$$

one can write

$$\begin{aligned} l(m_1, \dots, m_n) &= \prod_{i=1}^n E[\psi'^2(\theta_i) / \mathcal{F}_{m_i}] - \prod_{i=1}^n E\left[\psi'^2(\theta_i) - \frac{\psi''(\theta_i)}{m_i + r_i} / \mathcal{F}_{m_i}\right] \\ &= \prod_{i=1}^n E[\psi'^2(\theta_i) / \mathcal{F}_{m_i}] - E\left[\prod_{i=1}^n \left(\psi'^2(\theta_i) - \frac{\psi''(\theta_i)}{m_i + r_i}\right) / \mathcal{F}_{m_1, \dots, m_n}\right] \\ &= E\left[\sum_{i=1}^n \frac{\psi''(\theta_i)}{m_i + r_i} \cdot \prod_{j \neq i} \psi'^2(\theta_j) / \mathcal{F}_{m_1, \dots, m_n}\right] + E[F / \mathcal{F}_{m_1, \dots, m_n}] \end{aligned}$$

where F is an algebraic sum of terms c_{k_1, k_2} with c_{k_1, k_2} is a product of k_1 terms from the sequence $\left(\frac{\psi''(\theta_i)}{m_i + r_i}\right)_{i=1, \dots, n}$ and k_2 terms from the sequence $(\psi'^2(\theta_j))_{j=1, \dots, n}$, $k_1 \geq 2$ and $k_1 + k_2 = n$.

One assumes that there exists $p \geq 1$ such that

$$E[(\psi''(\theta_i))^p] < +\infty \text{ and } E[(\psi'(\theta_i))^{2p}] < +\infty \quad (3)$$

for all $i = 1, \dots, n$; then the corresponding Bayes risk associated with loss and cost can be written as

$$R(m_1, \dots, m_n) = E \left[\sum_{i=1}^n \frac{U_{im_i}}{m_i + r_i} + \sum_{i=1}^n c_i m_i \right] + \sum_{i=1}^n o\left(\frac{1}{m_i}\right)$$

and by the way, approximated for large samples by

$$\tilde{R}(P) = \tilde{R}(m_1, \dots, m_n) = E \left[\sum_{i=1}^n \frac{U_{im_i}}{m_i + r_i} + \sum_{i=1}^n c_i m_i \right]$$

where $U_{im_i} = E[V_i / \mathcal{F}_{m_1, \dots, m_n}]$ and $V_i = \psi''(\theta_i) \prod_{j \neq i} \psi'^2(\theta_j)$

3. Lower bound for the scaled Bayes risk

Theorem 1. *For any policy P , the following inequality holds:*

$$\frac{\tilde{R}(P)}{\sqrt{\sum_{j=1}^n c_j}} \geq 2E \left[\sum_{i=1}^n \sqrt{\frac{c_i}{\sum_{j=1}^n c_j}} \sqrt{U_{im_i}} \right] - \sum_{i=1}^n \sqrt{\frac{c_i}{\sum_{j=1}^n c_j}} \sqrt{c_i r_i}$$

PROOF. The proof is a direct consequence of the following inequality:

$$\begin{aligned} \frac{U_{im_i}}{m_i + r_i} + c_i m_i &= \frac{U_{im_i}}{m_i + r_i} + c_i (m_i + r_i) - c_i r_i \\ &= \left(\frac{\sqrt{U_{im_i}}}{\sqrt{m_i + r_i}} + \sqrt{c_i} \sqrt{m_i + r_i} \right)^2 + 2\sqrt{c_i} \sqrt{U_{im_i}} - c_i r_i \\ &\geq 2\sqrt{c_i} \sqrt{U_{im_i}} - c_i r_i. \end{aligned}$$

From now on, the notation $c \rightarrow 0$ means that $c_j \rightarrow 0$, for all $j = 1, \dots, n$. Furthermore, assume that for all i ,

$$\frac{c_i}{\sum_{j=1}^n c_j} \rightarrow \lambda_i \in]0, 1[, \text{ as } c \rightarrow 0.$$

Theorem 2. *For any random design P ,*

$$\liminf_{c \rightarrow 0} \frac{\tilde{R}(P)}{\sqrt{\sum_{j=1}^n c_j}} \geq 2E \left[\sum_{i=1}^n \sqrt{\lambda_i} \sqrt{V_i} \right]$$

PROOF. Theorem (1) and Fatou's lemma give immediately

$$\liminf_{c \rightarrow 0} \frac{\tilde{R}(P)}{\sqrt{\sum_{j=1}^n c_j}} \geq 2E \left[\liminf_{c \rightarrow 0} \sum_{i=1}^n \sqrt{\frac{c_i}{\sum_{j=1}^n c_j}} \sqrt{U_{im_i}} \right] = 2E \left[\sum_{i=1}^n \sqrt{\lambda_i} \sqrt{V_i} \right]$$

Theorem 3. *For any design satisfying*

$$m_i \sqrt{c_i} \rightarrow a_i \neq 0, \text{ a.e., as } c \rightarrow 0;$$

then

$$\liminf_{c \rightarrow 0} \frac{R(P)}{\sqrt{\sum_{j=1}^n c_j}} = \liminf_{c \rightarrow 0} \frac{\tilde{R}(P)}{\sqrt{\sum_{j=1}^n c_j}} \geq 2E \left[\sum_{i=1}^n \sqrt{\lambda_i} \sqrt{V_i} \right]$$

PROOF. Since

$$\liminf_{c \rightarrow 0} \frac{R(P)}{\sqrt{\sum_{j=1}^n c_j}} = \liminf_{c \rightarrow 0} \frac{\tilde{R}(P)}{\sqrt{\sum_{j=1}^n c_j}} + \lim_{c \rightarrow 0} \frac{1}{\sqrt{\sum_{j=1}^n c_j}} \sum_{i=1}^n o\left(\frac{1}{m_i}\right)$$

and

$$\lim_{c \rightarrow 0} \frac{1}{\sqrt{\sum_{j=1}^n c_j}} \sum_{i=1}^n o\left(\frac{1}{m_i}\right) = \lim_{c \rightarrow 0} \sum_{i=1}^n \frac{\sqrt{c_i}}{m_i \sqrt{c_i} \sqrt{\sum_{j=1}^n c_j}} \frac{o\left(\frac{1}{m_i}\right)}{\frac{1}{m_i}} = 0$$

then the proof follows.

4. First order optimal design

We look for designs satisfying

$$\frac{\tilde{R}(P)}{\sqrt{\sum_{j=1}^n c_j}} - 2E \left[\sum_{i=1}^n \sqrt{\lambda_i} \sqrt{V_i} \right] \rightarrow 0, \text{ as } c \rightarrow 0.$$

So, we write

$$\begin{aligned} & \frac{\tilde{R}(P)}{\sqrt{\sum_{j=1}^n c_j}} - 2E \left[\sum_{i=1}^n \sqrt{\lambda_i} \sqrt{V_i} \right] = \\ & \frac{2E \left[\sum_{i=1}^n \sqrt{c_i} \sqrt{U_{im_i}} \right]}{\sqrt{\sum_{j=1}^n c_j}} - 2E \left[\sum_{i=1}^n \sqrt{\lambda_i} \sqrt{V_i} \right] \\ & + \frac{E \left[\sum_{i=1}^n \frac{(\sqrt{U_{im_i}} - (m_i + r_i) \sqrt{c_i})^2}{m_i + r_i} \right]}{\sqrt{\sum_{j=1}^n c_j}} - \sum_{i=1}^n \sqrt{c_i} \sqrt{\frac{c_i}{\sum_{j=1}^n c_j} r_i} \end{aligned}$$

Hence, sufficient conditions for a such design to be asymptotically convergent are:

$$2E \left[\sum_{i=1}^n \sqrt{\frac{c_i}{\sum_{j=1}^n c_j}} \sqrt{U_{im_i}} \right] - 2E \left[\sum_{i=1}^n \sqrt{\lambda_i} \sqrt{V_i} \right] \rightarrow 0 \quad (4)$$

$$E \left[\frac{(\sqrt{U_{im_i}} - (m_i + r_i) \sqrt{c_i})^2}{(m_i + r_i) \sqrt{\sum_{j=1}^n c_j}} \right] \rightarrow 0, \forall i \quad (5)$$

as $c \rightarrow 0$.

Theorem 4. *Let P a random policy satisfying $m_i \rightarrow +\infty$, a.e. Suppose that condition (3) is true, then*

$$E \left[\sum_{i=1}^n \sqrt{\frac{c_i}{\sum_{j=1}^n c_j}} \sqrt{U_{im_i}} \right] - E \left[\sum_{i=1}^n \sqrt{\lambda_i} \sqrt{V_i} \right] \rightarrow 0, \text{ as } c \rightarrow 0.$$

PROOF. Remark that

$$\lim_{m_1, \dots, m_n \rightarrow +\infty} \sqrt{U_{im_i}} = \sqrt{V_i}, \text{ a.e.} \quad (6)$$

Now

$$\begin{aligned} \sup_{m_1, \dots, m_n} E \left[\left(\sqrt{U_{im_i}} \right)^2 \right] &= \sup_{m_1, \dots, m_n} E [U_{im_i}] \\ &= E \left[\psi''(\theta_i) \prod_{j \neq i} \psi'^2(\theta_j) \right] \\ &= E \left[\psi''(\theta_i) \right] \prod_{j \neq i} E \left[\psi'^2(\theta_j) \right] < +\infty \end{aligned}$$

hence, the uniform integrability of $\sqrt{U_{im_i}}$ follows from condition (3) and martingales properties. Therefore, the convergence in (6) holds in L^1 and consequently :

$$\sqrt{\frac{c_i}{\sum_{j=1}^n c_j}} \sqrt{U_{im_i}} \rightarrow \sqrt{\lambda_i} \sqrt{V_i} \text{ in } L^1, \text{ as } c \rightarrow 0,$$

which achieves the proof.

5. The two stage procedure

Following the previous section, our strategy now is to satisfy condition (5) as $c \rightarrow 0$. Then, we define the two stage sequential scheme as follows.

Stage one proceed for k_i observation from population P_i for $i = 1, \dots, n$; such that $k_i \sqrt{c_i} \rightarrow 0$ and $k_i \rightarrow +\infty$ as $c_i \rightarrow 0$.

Stage two select m_i integer as follows :

$$m_i = \max \{k_i, \hat{m}_i\}$$

where

$$\hat{m}_i = \left\lceil \frac{\sqrt{U_{ik_i}}}{\sqrt{c_i}} - r_i \right\rceil$$

and

$$U_{ik_i} = E \left[\psi''(\theta_i) \prod_{j \neq i} \psi'^2(\theta_j) / \mathcal{F}_{k_1, \dots, k_n} \right]$$

We give now the main result of the paper:

Theorem 5. *Assume condition (3) satisfied for a $p \geq 1$, then the two stage design is first order optimal.*

PROOF. The m_i , as defined by the two stage, satisfies

$$\lim_{c_i \rightarrow 0} (m_i + r_i) \sqrt{c_i} = \sqrt{V_i}$$

and since

$$\sqrt{\sum_{j=1}^n c_j} (m_i + r_i) = \frac{\sqrt{c_i} (m_i + r_i)}{\sqrt{\sum_{j=1}^n c_j}} \rightarrow \sqrt{\frac{V_i}{\lambda_i}}, \text{ as } c \rightarrow 0,$$

then :

$$\frac{(\sqrt{U_{im_i}} - (m_i + r_i) \sqrt{c_i})^2}{\sum_{j=1}^n c_j (m_i + r_i)} \rightarrow 0, \text{ a.e., as } c \rightarrow 0 \quad (7)$$

To show the convergence in L^1 , it will be sufficient to show the uniform integrability of the left hand side of (7).

Now, observe that:

$$\begin{aligned}
\frac{(\sqrt{U_{im_i}} - (m_i + r_i) \sqrt{c_i})^2}{\sqrt{\sum_{j=1}^n c_j (m_i + r_i)}} &\leq \frac{U_{im_i} + (m_i + r_i)^2 c_i}{\sqrt{\sum_{j=1}^n c_j (m_i + r_i)}} \\
&\leq \frac{U_{im_i}}{\sqrt{\sum_{j=1}^n c_j (m_i + r_i)}} + (m_i + r_i) \sqrt{c_i} \sqrt{\frac{c_i}{\sum_{j=1}^n c_j}} \\
&\leq \frac{U_{im_i}}{\sqrt{U_{ik_i}}} + \sqrt{U_{ik_i}}
\end{aligned}$$

and the uniform integrability will be established if one shows that $U_{im_i}/\sqrt{U_{ik_i}}$ and $\sqrt{U_{ik_i}}$ are both uniformly integrable. So, $\sqrt{U_{ik_i}}$ is uniformly integrable as a result of martingales L^p convergence properties with $p = 2$. To prove the uniform integrability of $U_{im_i}/\sqrt{U_{ik_i}}$ remark that

$$\frac{U_{im_i}}{\sqrt{U_{ik_i}}} \leq \max_{k'} \sqrt{U_{ik'}}$$

and for the remainder of the proof, we use Doob's inequality to show that

$$E \left[\max_{k'} \sqrt{U_{ik'}} \right] < +\infty$$

We have,

$$E \left[\max_{k'} \left(\sqrt{U_{ik'}} \right)^{2p} \right] \leq \left(\frac{2p}{2p-1} \right)^{2p} E \left[\left(\sqrt{V_i} \right)^{2p} \right] < +\infty$$

hence, since $p \geq 1$, $\max_{k'} \sqrt{U_{ik'}}$ is integrable and the proof follows.

6. Conclusion

The proof of the first order asymptotic optimality for the two stage design has been obtained mainly through an adequate scaling of the approximated Bayes risk associated with squared error loss and cost, a lower bound for the scaled Bayes risk independent of allocation, martingales properties and Doob's inequality.

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